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Noncommutative field theory and violation of translation invariance

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Abstract: Noncommutative field theories with commutator of the coordinates of the form $[x^\mu, x^\nu] = i\,\Lambda^{\mu\nu}{}_\omega x^\omega$ with nilpotent structure constants are studied and shown that a free quantum field theory is not affected. Invariance under translations is broken and the conservation of energy-momentum is violated, obeying a new law which is expressed by a Poincaré-invariant equation. The resulting new kinematics is studied and applied to simple examples and to astrophysical puzzles, such as the observed violation of the GZK cutoff. The $\lambda\Phi^4$ quantum field theory is also considered in this context. In particular, self interaction terms violate the usual conservation of energy-momentum and, hence, the radiative correction to the propagator is altered. The correction to first order in λ is calculated. The usual UV divergent terms are still present, but a new type of term also emerges, which is IR divergent, violates momentum conservation and implies a correction to the dispersion relation.

Keywords: Space-Time Symmetries, Non-Commutative Geometry.

Dedicated to the memory of Luís Guisado.

*Deceased.

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Contents

1. Introduction

Noncommutativity of coordinates has been intensively studied in the literature as it arises in the context of string theory [1], but also because it has interesting properties and implications in field theory [2, 3]. In many treatments of noncommutative field theory the noncommutative parameters are not regarded as Lorentz tensors, but instead a set of numbers that do not transform covariantly which implies naturally in the breaking of Lorentz invariance down to the stability subgroup of the noncommutative parameter [4]. Furthermore, noncommutative structures of the Lie-type imply in the violation of the energy-momentum conservation as the coordinate commutation relations break translational invariance. Alternatively, one could consider instead the noncommutative parameter as a Lorentz tensor covariant under Lorentz boosts. This approach has been studied earlier and in this framework we have shown that a noncommmutative scalar field coupled to gravity admits a covariant formulation (where associativity[1] is mantained only at perturbative level) which is compatible with a homogeneous and isotropic space-time [6]. Further attempts along these lines include work on noncommutative scalar field theory in three-dimensions [7] and on QED [8]. We mention that in the latter, gauge invariance is implemented via the introduction of a noncommutative gauge field, a "star" gauge invariance and "star" commutators.

[1]The relation between a nonassociative star product on D-branes and noncommutative theories on curved spaces has been discussed in [5].

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In this work we shall analyse classical and quantum field theory features of models where the noncommutativity of the coordinates has the following form

$$[x^{\mu}, x^{\nu}] = i\,\Lambda^{\mu\nu}{}_{\omega} x^{\omega}\,, \tag{1.1}$$

with the condition of nilpotency as specified below. This leads to a violation of the symmetry under translations and, consequently, requires a reformulation of energy-momentum conservation. This reformulation is proven to be Poincaré-invariant and reduces to the usual momentum conservation in the commutative limit. The formalism we develop follows the study of ref. [9], where the Baker-Hausdorff formula is related to the Kontsevich (see [10] and ealier references therein) noncommutative product, for Lie algebras of the form (1.1).

Before closing our introduction, let us point out that our nilpotency condition (cf. eq. (2.13) below) excludes noncommutative structures of the Lie-type such as the semisimple Lie algebras (SU(2) for the fuzzy sphere) and the κ-deformed Minkowsky space that are often discussed in the literature (see ref. [11] for an extensive review). Furthermore, we mention that recent work by Robbins and Sethi [12] is closely related with ours, even though it considers examples that are more directly inspired by string theory.

2. Mathematical formulation

2.1 Noncommutative algebra

A noncommutative associative product may be defined through the Lie-algebra commutator eq. (1.1), where $\Lambda^{\mu\nu\omega}$ is a real tensor with units of mass^{-1} and $\Lambda^{\mu\nu\omega} = -\Lambda^{\nu\mu\omega}$. On its hand, associativity implies the Jacobi identity

$$\Lambda^{\mu\nu}{}_{\omega}\Lambda^{\omega\alpha}{}_{\beta} + \Lambda^{\nu\alpha}{}_{\omega}\Lambda^{\omega\mu}{}_{\beta} + \Lambda^{\alpha\mu}{}_{\omega}\Lambda^{\omega\nu}{}_{\beta} = 0\,. \tag{2.1}$$

A noncommutative Fourier mode is defined by

$$e_*^{ik\cdot x} = \sum_{n=0}^{\infty}\frac{i^n}{n!}\overbrace{(k\cdot x)*\cdots*(k\cdot x)}^{n\text{ factors}} = \sum_{n=0}^{\infty}\frac{i^n}{n!}\,(k\cdot x)_*^n\,, \tag{2.2}$$

and we study the functional space spanned by these Fourier modes, with elements of the form

$$f(x) = \int\frac{d^nk}{(2\pi)^n}\tilde{f}(k)\,e_*^{ik\cdot x} \tag{2.3}$$

which, in the commutative limit, reduces to the usual Hilbert space. Notice that in eq. (2.2) the star product acts only on the configuration variables and not on the momentum ones.

The product of two generic functions is then given by

$$f*g = \int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\tilde{f}(k)\,\tilde{g}(q)\,e_*^{ik\cdot x}*e_*^{iq\cdot x}\,, \tag{2.4}$$

where we have expressed the functions in terms of their noncomutative Fourier expansion. This product is completely determined if the product of two Fourier modes $e_*^{ik\cdot x} * e_*^{iq\cdot x}$ can be evaluated. This can be achieved by making use of the Baker-Hausdorff formula

$$e_*^{ik\cdot x} * e_*^{iq\cdot x} = \exp_* \left\{ i\left(k+q\right)\cdot x + \frac{1}{2}\left[ik\cdot x, iq\cdot x\right] + \cdots \right\}, \tag{2.5}$$

where the dots stand for higher order commutators. Since the commutators obey

$$[x^{\mu_1}, [x^{\mu_2}, \ldots, [x^{\mu_n}, x^{\nu}]] \cdots] \propto i^n x^{\omega}, \tag{2.6}$$

the product of two Fourier modes is a Fourier mode

$$e_*^{ik\cdot x} * e_*^{iq\cdot x} = e_*^{i[k+q+V(k,q)]\cdot x} \tag{2.7}$$

with V determined by the Baker-Hausdorff expansion:

$$V_\omega\left(k,q\right) = k_\mu q_\nu \Lambda^{\mu\nu}{}_{\lambda} \left[-\frac{1}{2}\delta^{\lambda}_{\omega} + \frac{k_\alpha - q_\alpha}{12}\Lambda^{\alpha\lambda}{}_{\omega}\right] + O(\Lambda^3)\,. \tag{2.8}$$

2.2 Quadratic actions

In order to build actions, a star-integration must be defined. In the functional space whose elements are of the form (2.3), any function can be integrated if the integral of a Fourier mode is known. Hence, we introduce the following star-integration

$$\int_* d^n x\, e_*^{ir\cdot x} = \left(2\pi\right)^n \delta\left(r\right), \tag{2.9}$$

which yields the usual integration in the commutative limit.

Consider now the star-integral

$$I = \int_* d^n x\, f * g\,. \tag{2.10}$$

which, in Fourier space, is written as

$$I = \int \frac{d^n k}{\left(2\pi\right)^n}\frac{d^n q}{\left(2\pi\right)^n}\tilde{f}\left(k\right)\tilde{g}\left(q-k\right)\int_* d^n x\, e_*^{i[q+V(k,q-k)]\cdot x}, \tag{2.11}$$

and implies

$$I = \int \frac{d^n k}{\left(2\pi\right)^n} d^n q \tilde{f}\left(k\right)\tilde{g}\left(q-k\right)\delta\left(q+V\left(k,q-k\right)\right). \tag{2.12}$$

If the structure constants are nilpotent, that is, for $n > n_*$

$$\Lambda^{\mu_1\nu}{}_{\omega_1}\Lambda^{\mu_2\omega_1}{}_{\omega_2}\cdots\Lambda^{\mu_n\omega_{n-1}}{}_{\omega_n} = 0\,, \tag{2.13}$$

then

$$\delta\left(q+V\left(k,q-k\right)\right) = \frac{\delta\left(q\right)}{\left|\det\left(\delta^{\mu}_{\nu} - \frac{\partial V_\nu}{\partial q_\mu}\right)\right|} = \delta\left(q\right) \tag{2.14}$$

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since $\det(1+M) = 1$ if $M^n = 0$, which holds if Λ is nilpotent. Thus

$$I = \int \frac{d^n k}{(2\pi)^n} \tilde{f}(k)\, \tilde{g}(-k) = \int d^n x\, f_C(x)\, g_C(x) \tag{2.15}$$

where f_C, g_C are inverse Fourier transforms *using commutative Fourier modes*

$$f_C(x) = \int \frac{d^n k}{(2\pi)^n} \tilde{f}(k)\, e^{ik\cdot x}\,. \tag{2.16}$$

Equation (2.15) states that, in momentum space, quadratic terms in the lagrangian are the same as their commutative counterparts. In particular, free propagators will remain unchanged.

3. Violation of momentum conservation

We have concluded that the quadratic part of a lagrangian is not changed and, hence, the free theory is the same as the commutative one. In particular, the free Green function is equal to the commutative case and the dispersion relation $\epsilon^2 = p^2 + m^2$ is unchanged, since it is given by the poles of the free propagator. Yet, we shall see that interactions are altered by non-commutativity.

Consider a non-commutative field theory, with generic fields A_i and an interaction term

$$S_I = \int_* d^n x\, M_{i_1 \ldots i_m} A_{i_1} * \cdots * A_{i_m}\,, \tag{3.1}$$

where $M_{i_1 \cdots i_m}$ are constants.

Writing the fields in momentum space we get

$$S_I = \int \left[\prod_{i=1}^{m} \frac{d^n k_i}{(2\pi)^n}\right] \tilde{M}_{i_1 \cdots i_m}(\underline{k}_m)\, \tilde{A}_{i_1}(k_1) \cdots \tilde{A}_{i_m}(k_m) \tag{3.2}$$

where we use the notation $\underline{k}_m = (k_1, \ldots, k_m)$. The interaction in momentum space is given by

$$\tilde{M}_{i_1 \cdots i_m}(\underline{k}_m) = M_{i_1 \cdots i_m} \int_* d^n x\, e_*^{ik_1\cdot x} * \cdots * e_*^{ik_m \cdot x}\,. \tag{3.3}$$

In eq. (3.2) the variables k_i are mute, so we can sum over all π permutations of the indices i_m:

$$S_I = \int \left[\prod_{i=1}^{m} \frac{d^n k_i}{(2\pi)^n}\right] \tilde{M}^{\text{symm}}_{i_1 \cdots i_m}(\underline{k}_m)\, \tilde{A}_{i_1}(k_1) \ldots \tilde{A}_{i_m}(k_m)\,, \tag{3.4}$$

where

$$\tilde{M}^{\text{symm}}_{i_1 \cdots i_m}(\underline{k}_m) = \frac{1}{m!} \sum_{\pi\ \text{perm.}} (-)^{N(\pi)}\, \tilde{M}_{i_{\pi(1)} \cdots i_{\pi(m)}}\left(\underline{k}_{\pi(m)}\right). \tag{3.5}$$

To evaluate eq. (3.3), we use the expression

$$e_*^{ik_1\cdot x} * \cdots * e_*^{ik_m\cdot x} = \exp_* \left\{ i \sum_{j=1}^{m} k_j \cdot x + i V^m(\underline{k}_m) \cdot x \right\} \tag{3.6}$$

where

$$V^m\left(\underline{k}_m\right)=V^{m-1}\left(\underline{k}_{m-1}\right)+V\left(\sum_{i=1}^{m-1}k_i+V^{m-1}\left(\underline{k}_{m-1}\right),k_m\right) \tag{3.7}$$

with $V^2\left(\underline{k}_2\right)=V\left(k_1,k_2\right)$. This yields both the noncommutative energy-momentum law and the noncommutative vertex

$$\tilde{M}_{i_1\cdots i_m}\left(\underline{k}_m\right)=\left(2\pi\right)^n\delta\left(\sum_{i=1}^{m}k_i+V^m\left(\underline{k}_m\right)\right)M_{i_1\cdots i_m}\,. \tag{3.8}$$

Hence, the new energy-momentum law for the vertex reads

$$\sum_{i=1}^{m}k_i+V^m\left(\underline{k}_m\right)=0\,. \tag{3.9}$$

The full theory involves $\tilde{M}^{symm}_{i_1\cdots i_m}$, which will have contributions whenever

$$\sum_{i=1}^{m}k_i+V^m\left(\underline{k}_{\pi(m)}\right)=0 \tag{3.10}$$

for all $m!$ permutations of indices, π.

Thus, we see that the energy-momentum conservation is violated as the theory is not invariant under translations. In fact, in a translation $x^\mu\to x^\mu+b^\mu$, the commutator of the coordinates is changed by

$$\left[x^\mu,x^\nu\right]\to i\,\Lambda^{\mu\nu}{}_{\omega}x^\omega+i\,\theta^{\mu\nu}\,, \tag{3.11}$$

that is, a constant term $\theta^{\mu\nu}=\Lambda^{\mu\nu}{}_{\omega}b^\omega$ is added to the commutator of the coordinates. So, the interaction vertex becomes

$$\tilde{M}_{i_1\cdots i_m}\left(\underline{k}_m\right)\to\left(2\pi\right)^n\delta\left(\sum_{i=1}^{m}k_i+V^m\left(\underline{k}_{\pi(m)}\right)\right)M_{i_1\cdots i_m}\exp\left\{i\theta^m\left(\underline{k}_m\right)\right\} \tag{3.12}$$

where

$$\theta^m\left(\underline{k}_m\right)=\theta^{m-1}\left(\underline{k}_{m-1}\right)+\theta\left(\sum_{i=1}^{m-1}k_i+V^{m-1}\left(\underline{k}_{m-1}\right),k_m\right) \tag{3.13}$$

and $\theta^2\left(\underline{k}_2\right)=k_{1\mu}\theta^{\mu\nu}k_{2\nu}$. Hence, the interaction vertex is altered by an overall oscillating momentum-dependent factor and, thus, invariance under translations is broken. This example shows that translations give always rise to a constant term in the noncomutative tensor. However, the new energy- momentum law is unchanged, so it is a Poincaré-invariant expression, even though the theory is not.

4. Kinematical applications

4.1 Preliminaries

The first non-trivial behaviour arising from the new interaction vertex occurs with three particles. The energy-momentum equation is found to be

$$k_1+k_2+k_3+V\left(k_1,k_2\right)+V\left(k_1+k_2+V\left(k_1,k_2\right),k_3\right)=0 \tag{4.1}$$

and similar expressions for all permutations of the indices.

On physical grounds, eq. (4.1) represents three particles interacting. Formally, these equations can be treated in the context of the usual momentum conservation if one thinks in terms of four interacting particles, with the fourth particle's energy-momentum vector being given by a nonlinear function of the others. This reasoning may be extended to the m-particle case, eq. (3.10).

For the time being, let us consider a simple model with

$$\Lambda^{\mu_1\nu}{}_{\omega_1}\Lambda^{\mu_2\omega_1}{}_{\omega_2} = 0 \tag{4.2}$$

which complies with the Jacobi identity eq. (2.1).

The energy-momentum equation becomes

$$k_1 + k_2 + k_3 + V(k_1, k_2) = 0\,, \tag{4.3}$$

where

$$V_\omega(k_1, k_2) = \frac{1}{2} k_{1\mu} k_{2\nu} \Lambda^{\mu\nu}{}_\omega\,. \tag{4.4}$$

There are nontrivial covariant solutions to eq. (4.2). For instance, consider a constant antisymmetric tensor $\Lambda^{\mu\nu} = -\Lambda^{\nu\mu}$ with nontrivial kernel, that is, $\det\Lambda = 0$, and a non-vanishing vector r^ω belonging to this kernel. Hence a solution is given by

$$\Lambda^{\mu\nu\omega} = \Lambda^{\mu\nu} r^\omega\,. \tag{4.5}$$

In four dimensions we can parametrize $\Lambda^{\mu\nu}$ with two spatial vectors $\vec{E}$ and $\vec{B}$

$$\Lambda^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad r_\nu = \begin{pmatrix} r_0 \\ r_x \\ r_y \\ r_z \end{pmatrix}. \tag{4.6}$$

Condition eq. (4.5) implies that

$$r^2 = |\vec{r}|^2 \left[\left(\frac{B}{E} \sin\delta \right)^2 - 1 \right], \tag{4.7}$$

with δ being the angle between $\vec{B}$ and $\vec{r}$. The massless, massive and tachyon regimes of V are readily identifiable. Since we assume that $\Lambda^{\mu\nu}$ is a Lorentz tensor, there are always inertial frames where $\vec{E}$ is non-vanishing, and the above expression holds only for such frames. If $B < E$ (a Lorentz-invariant inequality) then r^ω behaves like a tachyon; otherwise, the behaviour of r^ω will depend on δ.

From the momentum conservation, eq. (4.3), we get the following result

$$\Lambda^{\mu\nu}(k_1 + k_2 + k_3)_\nu = 0\,, \tag{4.8}$$

which states that the vector sum of the momenta belongs to the (nontrivial) kernel of the noncommutative tensor. We also have the following expressions

$$\Lambda^{\mu\nu} k_\nu = \begin{pmatrix} \vec{E}\cdot\vec{k} \\ -k_0\vec{E} + \vec{B}\times\vec{k} \end{pmatrix} \tag{4.9}$$

and

$$q_\mu \Lambda^{\mu\nu} k_\nu = \vec{E} \cdot \left(q_0 \vec{k} - k_0 \vec{q} \right) + \vec{B} \cdot \left(\vec{k} \times \vec{q} \right) . \tag{4.10}$$

Eqs. (4.8) and (4.9) imply that the three-momentum is conserved along the direction of $\vec{E}$. Energy is conserved if the total three-momentum $\sum \vec{k}_i$ is along the direction of $\vec{B}$. Also,

$$k_1 \Lambda k_2 = -k_1 \Lambda k_3 = k_2 \Lambda k_3 \tag{4.11}$$

and one is required only to study eq. (4.3) with $V(k_1, k_2)$ and $-V(k_1, k_2)$. Note that the second case is obtained by performing $\vec{E}, \vec{B} \to -\vec{E}, -\vec{B}$. Thus, once computations have been performed in the first case, the results in the second one are obtained by performing this substitution.

While performing calculations, the following dimensionless combinations of the masses and the noncommutative parameters arise:

$$x_i = \frac{1}{2} E \, |\vec{r}| \, m_i \, , \qquad y_i = \frac{1}{2} B s_\delta \, |\vec{r}| \, m_i \, . \tag{4.12}$$

where the notation $s_\omega = \sin \omega$ is used.

4.2 Massive particle decay

Consider now the decay of a massive particle Φ_3 into two particles Φ_1 and Φ_2, that is

$$\Phi_3 \to \Phi_1 + \Phi_2 \, . \tag{4.13}$$

Let m_i be the mass of particle Φ_i and $m_1 \geq m_2$. In the rest-frame of Φ_3 the angle α between particles Φ_1 and Φ_2 is given by

$$\cos \alpha = -\frac{1 + x_3 c_\theta c_\varphi}{\sqrt{(1 + x_3 c_\theta c_\varphi)^2 + (x_3 c_\theta s_\varphi)^2}} \tag{4.14}$$

where θ is the angle between $\vec{p}_1$ and $\vec{E}$, and φ the angle between $\vec{p}_1$ and $\vec{r}$. Also, the notation $c_\omega = \cos \omega$ is used. The absolute value of the right-hand side of this equation is always smaller than one, meaning that this decay is always possible. In the high energy regime ($x_3 \gg 1$) the variable x_3 decouples and one obtains $\alpha \approx \pm \varphi$ or $\alpha \approx \pi \pm \varphi$. In the low-energy regime ($x_3 \ll 1$) one finds the first-order correction $\alpha \approx \pi \pm x_3 s_\varphi c_\theta$.

The new equation for the energy is

$$m_3 = \epsilon_1 \left(1 - y_3 c_\theta v_1 \right) + \epsilon_2 \, , \tag{4.15}$$

where we have used ϵ_i, v_i as the energy and velocity of particle i.

If $|y_i| < 1$ then spontaneous decay will occur if m_3 satisfies the condition

$$m_3 > \frac{m_1 + m_2}{1 + y_1 c_\theta v_1} \, . \tag{4.16}$$

If the denominator is zero or negative, then the decay is impossible. We can see that if the velocity of particle 1 is high and mainly along the direction of $\vec{E}$ then the decay can occur for a value of m_3 smaller than the sum of the masses $m_1 + m_2$

$$m_3 > \frac{m_1 + m_2}{1 + |y_1|} \, . \tag{4.17}$$

4.3 Massless particle decay

The decay of a massless particle into two massive particles is kinematically forbidden. However, in the present model, all massless particles become unstable and can decay into two massive particles. If the particle Φ_3 is massless we have the following condition for the energy of the photon in the rest frame of particle Φ_1

$$\omega > \frac{2}{|Bs_\delta - Ec_\varphi||\vec{r}||c_\theta|} \equiv \frac{\omega_0(\varphi)}{|c_\theta|}\,, \tag{4.18}$$

where we have defined θ as the angle between $\vec{E}$ and $\vec{\omega}$ and also φ as the angle between $\vec{r}$ and $\vec{\omega}$. If the denominator is zero, the decay is impossible. Note that this limit is independent of the mass of the decaying particles. This result is only valid in the low-energy limit where $|x_1| < 1$, that is, when the decay produces particles with low mass.

The above limit for angle θ implies that

$$c_\theta^2 > \left(\frac{\omega_0}{\omega}\right)^2, \tag{4.19}$$

which states that, as the energy of the photon grows larger, the decay is possible for a wider range of θ angles. If $\theta = 0$, the decay is impossible.

4.4 The GZK cutoff

The Greisen-Zatsepin-Kuz'min (GZK) cutoff mechanism asserts that ultra-high-energy (UHE) protons with energies $\epsilon_p > 4 \times 10^{19}\, eV$ from sources beyond $50 - 100\, Mpc$ should not be observed, due to their interaction with Cosmic Microwave Background (CMB) photons. It has been proposed (for brief review see ref. [13]) that Lorentz-violating terms in the kinematics of hadronic reactions may be the answer to this puzzle. The GZK cutoff has the following dominant resonance

$$p + \gamma_{CMB} \to \Delta_{1232}\,. \tag{4.20}$$

It is easily shown that the model eq. (4.5) does not account for a violation of the GZK cutoff. In fact, in the case of head-on collision, the new equation for the energy yields

$$\epsilon_p[1 - y_\omega c_\theta(1 + v_p)] + \omega = \epsilon_\Delta\,, \tag{4.21}$$

where ϵ_i, v_i denote the energy and velocity of particle i and θ is the angle between $\vec{E}$ and $\vec{\omega}$. In order to occur any appreciable deviation that renders this reaction impossible (for instance $\epsilon_\Delta < m_\Delta$), one should have $y_\omega \approx 1$, which, given the low energy of the CMB photon, would yield a very small mass for the noncommutative parameters.

Nevertheless, the violation of the GZK limit may be explained in the context of the model

$$\Lambda^{\mu_1\nu}{}_{\omega_1}\Lambda^{\mu_2\omega_1}{}_{\omega_2}\Lambda^{\mu_3\omega_2}{}_{\omega_3} = 0\,, \tag{4.22}$$

with $\Lambda^{\mu_1\nu}{}_{\omega_1}\Lambda^{\mu_2\omega_1}_{\omega_2} \neq 0$. This cannot be implemented by model eq. (4.5), which complicates the analysis. The equation for the momentum is given by

$$(k_1 + k_2 + k_3)_\omega = k_{1\mu}k_{2\nu}\Lambda^{\mu\nu}{}_\lambda \left[-\frac{1}{2}\delta^\lambda_\omega + \frac{(k_1 - k_2)_\alpha}{12}\Lambda^{\alpha\lambda}{}_\omega\right] \tag{4.23}$$

where we have used eq. (4.1) recursively and the fact that cubic terms in Λ vanish.

This condition can be modeled by a simpler one, more suitable for phenomenological considerations which, however, breaks Lorentz invariance. As we have seen, the quadratic term in the momentum does not account for the violation of the GZK cutoff, so it will be dropped. Taking into account that the proton has the highest energy and the Δ the second highest energy, we can write the new momentum equation for the reaction (4.20) as

$$(k_p + k_\gamma)^\mu = k_\Delta^\mu - s^\mu \frac{\epsilon_p^2}{M^2}\epsilon_\Delta \tag{4.24}$$

where adimensional vector s^μ is of the order of unity and M is the typical noncommutative mass scale. In this case, the process is impossible if $s^0 > 0$ and $\epsilon_p > M$, which sets the scale of noncommutativity.

Note that one must consider all permutations of the indices in eq. (4.23). Due to the low energy of CMB photons, the permutations that lead to a term of the type ϵ_γ^2 will not violate the GZK cutoff. Since there are six permutations of the indices and only two lead to this type of term, we can estimate that 2/3 of the events leading to the ressonance (4.20) will violate the GZK cutoff.

It is generally believed [13, 14] that a cubic term in the equations of dispersion will explain the violation of this cutoff. In fact, eq. (4.24) can be obtained by assuming the usual momentum conservation and postulating a new equation of dispersion by the substitution

$$k^\mu \to k^\mu + s^\mu \frac{\epsilon^2}{M^2}\lambda \tag{4.25}$$

where λ represents the typical energy of the product of the reaction. This will lead to the following dispersion relation

$$m^2 = \epsilon^2 - p^2 + 2s^\mu v_\mu \frac{\lambda}{M^2}\epsilon^3 \tag{4.26}$$

where $v^\mu = (1, \vec{v})$ is the four-vector velocity, which we assume to be nearly light-speed. Only the lower order terms of the correction were kept.

Thus, it is as if a cubic term is added to the dispersion relation. However, this model differs from the one of ref. [14] in the sense that eq. (4.26) is sensible to the typical energy of the product of the reaction, that is, to the process in question. Also, there is a dependence on the geometry of the propagation of the particle, through the term $s^\mu v_\mu$. In addition, since the free theory is not altered by our approach, this effective dispersion equation may only be used to study particle reactions and not classical free-particle propagation.

5. Quantum field theory

To study the quantum aspects of the noncommutative model discussed in sections II and III we consider the $\lambda\Phi^4$ theory. The action is given by

$$S = \int_* d^n x \left[\frac{1}{2}\partial_\mu \Phi * \partial^\mu \Phi + \frac{m^2}{2}\Phi * \Phi + \frac{\lambda}{4!}\Phi * \Phi * \Phi * \Phi\right] \tag{5.1}$$

and can be evaluated in Fourier space to yield the corresponding Feynman rules. The vertices are already calculated in eq. (3.8) and the propagators are the same as the commutative ones. We shall consider the euclidean formulation.

The free propagator obeys the usual energy-momentum conservation. However, since interactions do not, it is expected that the quantum corrections to the propagator due to the self-interaction term in the action will not obey the usual momentum conservation. Therefore, it is of interest to compute the first order correction to the two-point function, which is proportional to

$$\Gamma_1 (p_i, p_f) = -\frac{1}{2}\frac{\lambda}{4!}\frac{1}{p_i^2+m^2}\frac{1}{p_f^2+m^2} I (p_i, p_f) \tag{5.2}$$

with

$$I (p_i, p_f) = \int \frac{d^n q}{(2\pi)^n} \frac{\sum_\pi \delta\left[p_i + p_f + V^4 (p_i, p_f, q, -q)\right]}{q^2+m^2}\,, \tag{5.3}$$

where the sum is computed over all π permutations of the arguments of V^4.

We proceed by evaluating this integral in the simple model eq. (4.2). From the several contributions to the above integral, there are two which differ from the commutative case. They are formally identical, and the relevant integrals are given by

$$J (p_i, p_f) = \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2+m^2} \delta[p_i + p_f + 2V (q, k)] \tag{5.4}$$

where $k = p_i, p_f$ for each case.

Using a Schwinger parametrization and the usual Fourier representation for the delta-function, the integrals are gaussian and yield

$$J (p_i, p_f) = \frac{1}{(4\pi)^n \sqrt{\det N}} \frac{1}{(p_i + p_f)\cdot N^{-1} \cdot (p_i + p_f) + m^2}\,, \tag{5.5}$$

where

$$N_{\omega\lambda} (k) = k_\nu \Lambda^{\mu\nu}{}_\omega k_\beta \Lambda^{\ \beta}_{\mu\ \lambda}\,. \tag{5.6}$$

Notice that this matrix is singular for model eq. (4.5), but not in general. The fact that Λ is nilpotent is not important, as $\Lambda^{\mu\nu}{}_\alpha$ is nilpotent only regarding the indices μ and α, and N involves only α type indices. Also, matrix N is singular if $k = p_i = 0$ or $k = p_f = 0$ and, hence, there is an IR divergence, which is usual in noncommutative quantum field theories.

The integral

$$J_C (p_i, p_f) = \delta (p_i + p_f) \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2+m^2}\,, \tag{5.7}$$

which is ultraviolet divergent for $n = 4$, also arises from eq. (5.3). Thus one concludes that its regularization is still required, although this is not necessary in eq. (5.4). Hence, the UV renormalizability properties at the one-loop approximation are not altered.

From the correction to the two-point function in eq. (5.5) two interesting features arise. First, the conservation of momentum is lost, since the delta function $\delta (p_i + p_f)$ is no longer present. Second, the correction to the dispersion relation, which is given by the pole of eq. (5.5), manifests itself in a quite specific way, involving the Lorentz algebra of the momentum vectors and matrix N^{-1}. In fact, the poles of eq. (5.5) suggest the particle is subjected to a momentum-dependent metric given by N^{-1}.

6. Discussion and conclusions

In this work we have presented a noncommutative field theory where the coordinates have a Lie-algebra commutator as eq. (1.1) with nilpotent structure constants. This breaks Lorentz as well as translational invariance. Free theory is unchanged so the propagators and the dispersion relations are not altered. The vertices show a new energy-momentum law, which steems from the breaking of translational invariance. The kinematical studies of such law where established in particle decay physics and shown how it can be applied as a possible explanation for the violation of the GZK cutoff, setting the noncommutative mass scale at $M \approx 4 \times 10^{19} eV$. A link between these kinematics and Lorentz-violating theories was established, using a simplified model. However, there are well definite differences between our approach and the ones usual discussed in the literature (see ref. [13]), the most important one being that an effective dispersion law always depends on the energy and geometry of the processes in question.

It is tempting to speculate that our results have a bearing on the other known astrophysical puzzles, namely the observation of high energy photons, $\epsilon \approx 20\, TeV$, from far away sources and the pion stability in extensive air showers (see ref. [13] and references therein). Indeed, since both phenomena can be understood via a cubic deformation in the relativistic dispersion relation, so at pair creation through the process $\gamma + \gamma_{IR\, bacground} \rightarrow e^+ + e^-$ cannot occur and pion decay into photons has a smaller width, it is plausible to assume that these paradoxes can be explained in our model as well.

In the context of quantum field theory, it was shown that it is possible to carry out explicit calculations regarding the first-order correction to the two-point function in $\lambda\Phi^4$ theory. The interaction terms violate momentum conservation and this is expressed in the two-point function, where the usual delta function structure $\delta\left(p_i + p_f\right)$ is lost. Even though the noncommutative contributions are UV finite, usual commutative integrals are still present and are UV divergent. Thus, the UV renormalization properties of one-loop calculations are not altered. Also, in a strict sense, it is shown that free theory is unchanged and so the propagators and the dispersion relations are not altered. New IR divergences arise in the noncommutative corrections, a feature which is shared with constant commutator noncommutative field theories, known as UV/IR mixing. The poles of the noncommutative terms indicate that there is a correction to the dispertion relation, through the Lorentz algebra of matrix N, eq. (5.6), which seems to indicate that the particle satifies a dispersion relation arising from a momentum-dependent metric, eq. (5.6).

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